Comment on "Tricritical Behavior in Rupture Induced by Disorder"

In their Letter [1], Andersen, Sornette, and Leung describe possible behaviors for rupture in disordered media. Their analysis is first based on the mean-field-like democratic fiber bundle model (DFBM). In the DFBM, N_0 ($\to \infty$) fibers are initially pulled with a force F, which is distributed uniformly. A fiber breaks if the stress it undergoes exceeds a threshold chosen from a probability distribution $p(x) = \frac{dP}{dx}$. After each set of failures, the force F is redistributed over the remaining fibers. Let $N(F)$ be the final number of intact fibers. What is the behavior of $N(F)$, as F increases from 0 to ∞? Reference [1] claims the existence of a tricritcal point, separating a "first-order" regime, characterized by an abrupt failure, from a "second-order" regime, characterized by a divergence in the breaking rate (response function) $N'(F)$.

Here, we present a graphical solution of the DFBM. Unlike an analytical solution, this enables us to consider the *qualitatively different classes* of disorder distribution, and to distinguish the corresponding *generic* behaviors of $N(F)$. We find that, for continuous distributions with finite mean, the system *always* undergoes a macroscopic failure, preceded by a diverging breaking rate. A "first-order" failure, with no preceding divergence, is an artifact of a (large enough) discontinuity in $p(x)$.

Suppose that a set of failures leaves the system with N_i unbroken fibers. Each of these is now under a stress F/N_i. This leads to another set of failures, bringing the number of intact fibers to $N_{i+1} = N_0\{1 - P(\frac{F}{N_i})\}$. The function $N(F)$, defined above, is nothing but N_∞. A graphical scheme for this iteration is facilitated by setting $x_i = N_i/F$, $f = F/N_0$, and $\pi(x) = 1 - P(1/x)$, leading to

$$f x_{i+1} = \pi(x_i)\,. \tag{1}$$

Since

$$\pi'(x) = \frac{1}{x^2}\, p\left(\frac{1}{x}\right), \tag{2}$$

$\pi(x)$ is a monotonic function of x, increasing from 0 to 1. Therefore, from iterating Eq. (1) graphically, $N(F)$ is given by the rightmost intersection of the curve $y = \pi(x)$ with the straight line $y = fx$. As the force is increased, the straight line becomes steeper, and the intersection consequently moves to the left.

We first consider continuous infinite-support distributions $p(x)$. We can distinguish three qualitatively different cases, depending on the behavior of $p(x)$ at large x (see Fig. 1). (i) *For* $p(x) \sim x^{-r}$, *with* $r < 2$, intact fibers remain at any force F. (ii) *For* $p(x) = \alpha x^{-2}$, $N(F)$ goes continuously to zero at $F_c = N_0\alpha$, with a diverging breaking rate $[N'(F_c) = \infty]$. In both cases, there may or may not be jumps in $N(F)$ at smaller forces. In particular, if

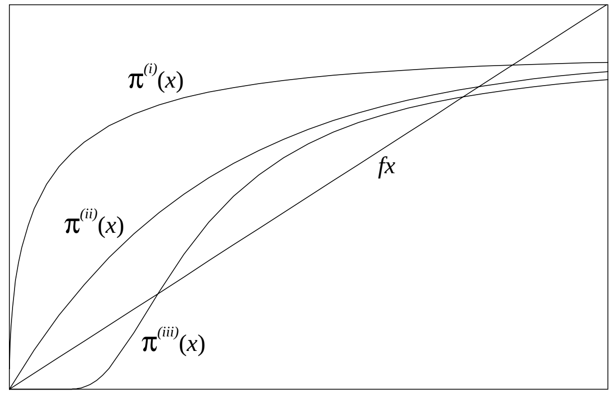

FIG. 1. Illustration of the graphical scheme for the three generic cases (i), (ii), and (iii) discussed in the text.

the slope of $\pi(x)$ is monotonically decreasing (π convex), $N(F)$ has no discontinuity. This is the case for, e.g., $p(x) = \frac{\alpha}{x^r}\, e^{-\alpha/x^{r-1}}$ $(1 < r \le 2)$, with any α. Note also that both classes of distributions yield an infinite mean $\langle x \rangle = \infty$. (iii) *For p(x) such that* $x^2 p(x) \to 0$, e.g., $p(x) = \frac{x}{\lambda^2}\, e^{-x/\lambda}$ or $p(x) = \frac{x}{\lambda}\, e^{-x^2/2\lambda}$ *with any* λ, there is at least one jump in $N(F)$, leading to $N = 0$. As seen graphically, any such jump is preceded by a diverging breaking rate, i.e., the curve $N(F)$ reaches its discontinuity vertically, generically according to $N'(F) \sim |F - F_c|^{-1/2}$.

From our graphical method, it appears clearly that an abrupt jump in $N(F)$ with no divergence preceding it is possible only if $\pi(x)$ has a nondifferentiable point, which in turn, by Eq. (2), requires a discontinuity in $p(x)$. We illustrate this in the context of finite-support distributions, for which $p(x) = 0$ for $x < a$ and $x > b$. No fiber breaks up to $F_a = N_0 a$. Then, as can be shown by the graphical method, an abrupt failure occurs at F_a only if $\pi'(1/a) \ge a$, which is equivalent to requiring a minimal discontinuity $p(a) \ge 1/a$ at a. Also, note that for any finite b, there will be at least one jump in $N(F)$, leading to $N = 0$. Thus, for the more physical case of a continuous—albeit finite-support—distribution, the behavior of the solution is identical to that of case (iii) above.

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[1] J. V. Andersen, D. Sornette, and K.-T. Leung, Phys. Rev. Lett. **78**, 2140 (1997).

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